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LETTER TO THE EDITOR

A test of conformal invariance: correlation functions on a disc

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Abstract. Using conformal invariance one can derive the correlation functions on a disc from those in the half-plane. The correlation function in the half-plane is determined by the 'small' conformal invariance up to an unknown function of one variable. By measuring, using the Monte Carlo method, the correlation function for two different configurations, the unknown function can be eliminated and one obtains a test of conformal invariance. It is shown that the Ising and the three-state Potts model pass the test for very small lattices.

It was shown by Cardy (1984) that at the critical temperature, the two-point correlation function in the half-plane x, y ($y \geq 0$) is determined by conformal invariance to have the expression:

$$G_{\mathrm{HP}}(y_1, y_2, x_1 - x_2) = (y_1 y_2)^{-\varkappa} \Phi(f) \tag{1}$$

where

$$f = [(x_1 - x_2)^2 + (y_1 - y_2)^2]/2y_1 y_2 \tag{2}$$

and

$$\lim_{f \to 0} \Phi(f) \sim f^{-\varkappa} \qquad \lim_{f \to \infty} \Phi(f) \sim f^{-\varkappa_s}. \tag{3}$$

Here $\varkappa$ ($\varkappa_s$) are the bulk (surface) scaling dimensions (Binder 1983). With the exception of the Ising model, the functions Φ are yet unknown. In the Ising case, for the spin–spin correlation one has (McCoy and Wu 1967)

$$\Phi = [(f+2)^{1/2} - f^{1/2}]^{1/2} [f(f+2)]^{-1/8}. \tag{4}$$

Using an analytic transformation (Belavin *et al* 1984) and equation (1) one can derive the correlation function for other geometries. This was done for the strip (Cardy 1984b); here we will consider the disc. When considering the strip one has the advantage that the correlation functions parallel to the strip are exponential and the exponent gives directly the scaling dimension $\varkappa_s$ (only the limit $f \to \infty$ of $\Phi(f)$ appears). The disadvantage is that one has to do the calculations using the transfer matrix and is thus limited to systems with few degrees of freedom. As we will see, in the disc geometry the full function $\Phi(f)$ appears but one has the advantage of a finite geometry and hence the possibility of using the Monte Carlo method (Metropolis *et al* 1953) and study systems with many degrees of freedom.

We write

$$z_j = x_j + \mathrm{i} y_j \qquad z_j = \rho_j \,\mathrm{e}^{\mathrm{i}\varphi_j} \qquad j = 1, 2 \tag{5}$$

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and perform the transformation

$$W(z) = U + \mathrm{i}V = R(\mathrm{i}z+1)/(\mathrm{i}z-1). \tag{6}$$

The transformation (6) takes the half-plane into a disc of radius R with free boundary conditions. Choosing $\varphi_1 = \varphi_2 = \frac{1}{2}\pi$ the two points $(0, y_1)$, $(0, y_2)$ are mapped into two points u_1, u_2 along a diameter of the disc:

$$u_j = \frac{y_j - 1}{y_j + 1}, \qquad -1 \leqslant u_j = \frac{U_j}{R} \leqslant 1, \qquad j = 1, 2. \tag{7}$$

The correlation function between the two points u_1 and u_2 is

$$\begin{aligned} G_{\mathrm{D}}(u_1, u_2) &= |W'(z_1)\, W'(z_2)|^{-x} G_{\mathrm{HP}}(y_1, y_2) \\ &= (2/R)^{2x}[(1-u_1^2)(1-u_2^2)]^{-x}\Phi(f) \end{aligned} \tag{8}$$

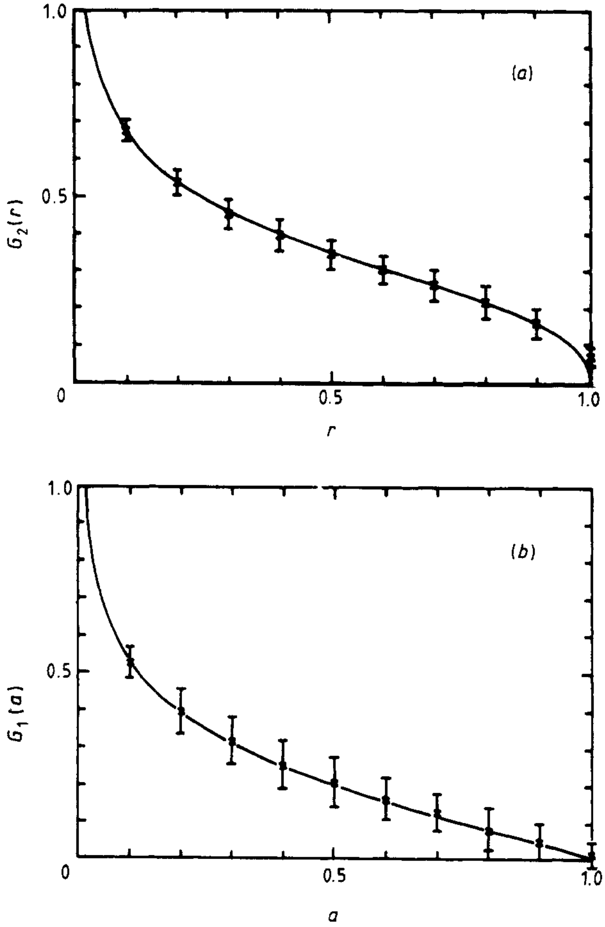

Figure 1. The spin–spin correlation functions of the Ising model on a disc of radius $R = 10$. (a) The function $G_2(r)$ (full curve) is obtained from equations (4) and (12). The error bars of the Monte Carlo data are statistical. (b) The function $G_1(a)$ (full curve) is obtained from equations (4) and (10).

where

$$f = 2(u_1 - u_2)^2/(u_1^2 - 1)(u_2^2 - 1). \tag{9}$$

If the function $\Phi(f)$ is known, the correlation function $G_D(u_1, u_2)$ on the disc is completely determined. This is the case for the Ising model. Since however in general $\Phi(f)$ is unknown, we can still use (8) to give a check of conformal invariance. Let us choose two configurations and use (8):

(a) $u_1 = -u_2 = a$, $(0 \leqslant a \leqslant 1)$

$$G_1(a) = G_D(-a, a) = (2/R)^{2x}(1 - a^2)^{-2x}\Phi(f) \tag{10}$$

with

$$f = 8a^2/(1 - a^2)^2. \tag{11}$$

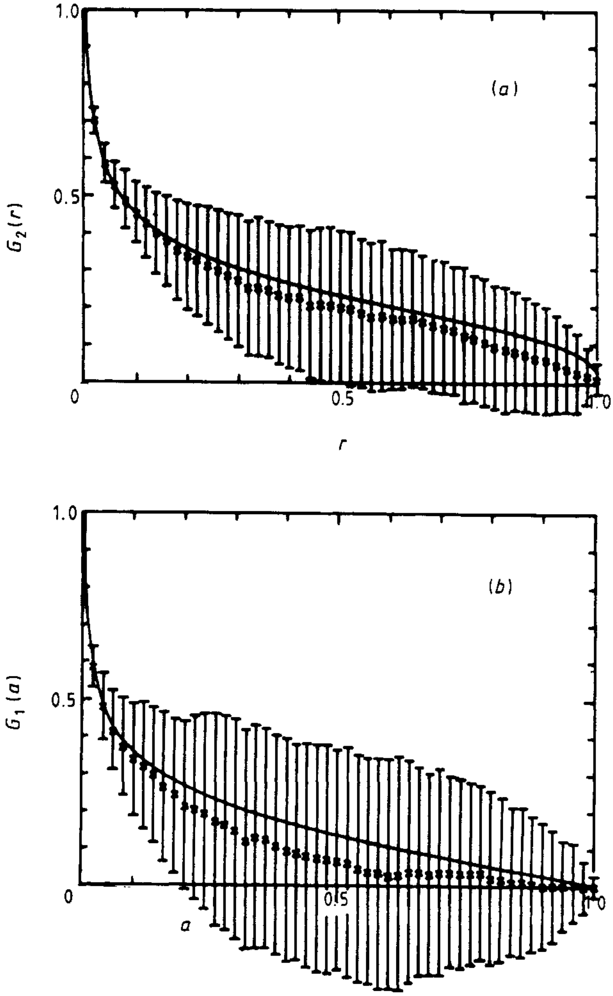

Figure 2. The spin-spin correlation functions of the Ising model on a disc of radius $R = 50$. (a) The function $G_2(r)$ (full curve) is obtained from equations (4) and (12). The error of the Monte Carlo data are statistical. (b) The function $G_1(a)$ (full curve) is obtained from equations (4) and (10).

(b) $u_1=0,\ u_2=r,\ (0\leq r\leq 1)$

$$G_2(r)=G_D(0,r)=(2/R)^{2x}(1-r^2)^{-x}\Phi(f) \tag{12}$$

with

$$f=2r^2/(1-r^2). \tag{13}$$

We now take

$$r=2a/(1+a^2) \tag{14}$$

such that the values of f in (11) and (12) are the same and obtain

$$G_2(r)/G_1(a)=(1+a^2)^{2x}. \tag{15}$$

We stress that the identity (15) is a consequence of conformal invariance alone and should apply also to systems where not all the representations of the conformal algebra are unitary (Belavin *et al* 1984, Friedan *et al* 1984). An example of such systems could be the Ashkin–Teller model.

We have tested the relation (15) for the Ising and the three-state Potts (1952) model. We have taken square lattices with $R=10, 15, 20, 25, 30, 35, 40, 45$ and 50 lattice spaces.

A disc lattice is constructed by the following procedure: draw a circle of radius R around the central point of a $(2R+1)\times(2R+1)$ square lattice and then drop all the points outside the circle, using free boundary conditions. The function $G_1(a)$ is measured for two points being equally spaced (a) from the central point in opposite directions. Then from one of these points and the central point we get two values for $G_2(r)$. For each radius we have performed at least 1600 sweeps for measurement and an appropriate number of sweeps for termination. The error bars shown in figures 1, 2 and 3 are derived from the statistical fluctuations.

We first consider the spin–spin correlation in the Ising model. Here the functions $G_2(r)$ and $G_1(a)$ are known since $\Phi(f)$ is known (see equation (4)). In figures 1 and 2 we show the measured values of $G(r)$ and $G(a)$ and compare them with the theoretical

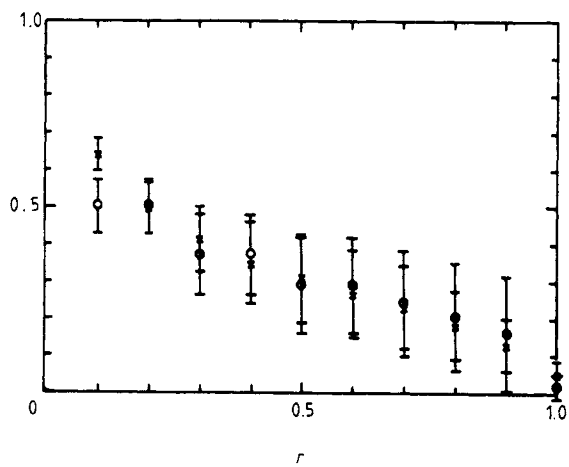

Figure 3. The spin–spin correlation functions of the three-state Potts model on a disc of radius $R=10$. ($\times$) represent the Monte Carlo values for $G_2(r)$, ($\bigcirc$) represent the Monte Carlo values for the function $G_1(a)(1+a^2)^{4/15}$. The errors are statistical.

values obtained from (10) and (12). We have normalised the theoretical curves at the Monte Carlo value for $r = 1/R$. We have represented the correlation functions for $R = 10$ (figure 1) and $R = 50$ (figure 2). We notice that for the small lattice ($R = 10$) the fluctuations are small and the agreement with the theoretical curves is excellent. The same is true for $R = 15$, 20 and 25. Beginning with $R = 30$, the fluctuations start to be large (the relaxation time increases too) and the agreement with the theoretical predictions is less spectacular.

We now consider the three-state Potts model. We have considered again the spin-spin correlation function ($\varkappa = \frac{2}{15}$) and use the test of conformal invariance given by equation (15). The functions $G_2(r)$ and $G_1(a)(1+a^2)^{4/15}$ are compared in figure 3 for $R = 10$. The average values of the correlation functions are close and we conclude, as expected, that at the critical point the system is conformal invariant.

In conclusion, when used on small lattices equation (15) provides us with a good check of conformal invariance and obviously one should test, using our method, other systems too.

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References

Ashkin J and Teller E 1943 *Phys. Rev.* **64** 178

Belavin A A, Polyakov A M and Zamolodchickov A B 1984 *Nucl. Phys.* B **241** 333

Binder K 1983 *Phase Transitions and Critical Phenomena* vol 8, ed C Domb and M S Green (London: Academic)

Cardy J L 1984a *Nucl. Phys.* B **240** 514

—— 1984b *J. Phys. A: Math. Gen.* **17** L385

Friedan D, Qui Z and Shenker S H 1984 *Phys. Rev. Lett.* **52** 1575

McCoy B and Wu T T 1967 *Phys. Rev.* **162** 436

Metropolis N, Rosenbluth A W, Rosenbluth N N, Teller A H and Teller E 1953 *J. Chem. Phys.* **21** 1087

Potts R B 1952 *Proc. Camb. Phil. Soc.* **48** 106